On stochastic fractional Volterra equations in Hilbert space

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February 2, 2008

Abstract

In this paper, stochastic Volterra equations, particularly fractional, in Hilbert space are studied. Sufficient conditions for mild solutions to be strong solutions are provided. Several examples of Volterra equations having strong solutions are given, as well.

1 Introduction

In this paper, which is the continuation of [7], we consider the following stochastic Volterra equation in a separable Hilbert space H

$$X(t) = X_0 + \int_0^t a(t-\tau)AX(\tau)d\tau + \int_0^t \Psi(\tau)\,dW(\tau)\;. \tag{1}$$

In (1), $X_0 \in H$, $a \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$, A is a closed unbounded linear operator in H with a dense domain $D(A)$ equipped with the graph norm $|\cdot|_{D(A)}$. W is a genuine Wiener process or a cylindrical Wiener process and Ψ is an appropriate process defined below.

2000 Mathematics Subject Classification: primary: 60H15, 60H20; secondary: 60H05, 45D05.
Key words and phrases: Stochastic fractional Volterra equation, α-times resolvent, strong solution, stochastic convolution,convergence of resolvent families.
The second author is partially financed by FONDECYT Grant #1050084

Equation (1) is motivated by a wide class of model problems and corresponds to an abstract stochastic version of several deterministic problems, mentioned, e.g. in [10] (see also the references therein).

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a stochastic basis and U a separable Hilbert space. Let $Q \in L(U)$ be a linear symmetric positive operator and $W(t),\ t \geq 0$, be an U-valued Wiener process with the covariance operator Q. Let us note that the noise term $Z(t) := \int_0^t \Psi(\tau)\, dW(\tau),\ t \geq 0$, includes two distinguish cases.

When $\mathrm{Tr}\, Q < +\infty$, hence $W(t),\ t \geq 0$, is a genuine Wiener process. Then we can take $U = H$, $\Psi := I$ and the noise term $Z(t)$ becomes $W(t),\ t \geq 0$.

If $\mathrm{Tr}\, Q = +\infty$, $W(t),\ t \geq 0$, is so-called cylindrical Wiener process. In this case, in order to provide a sense of the integral $Z(t)$, the process $\Psi(t),\ t \geq 0$, has to be an operator-valued process (see, e.g. [5]). We define the subspace $U_0 := Q^{1/2}(U)$ of the space U endowed with the inner product $\langle u, v\rangle_{U_0} := \langle Q^{-1/2}u, Q^{-1/2}v\rangle_U$.

By $L_2^0 := L_2(U_0, H)$ we denote the set of all Hilbert-Schmidt operators acting from U_0 into H; the set L_2^0 equipped with the norm $|C|_{L_2(U_0,H)} := \left(\sum_{k=1}^{+\infty} |Cu_k|_H^2\right)^{\frac{1}{2}}$, is a separable Hilbert space.

By $\mathcal{N}^2(0,T;L_2^0)$, where $T < +\infty$ is fixed, we denote a Hilbert space of all L_2^0-predictable processes Ψ such that $||\Psi||_T < +\infty$, where

$$||\Psi||_T := \left\{\mathbb{E}\left(\int_0^T |\Psi(\tau)|^2_{L_2^0}\, d\tau\right)\right\}^{\frac{1}{2}} = \left\{\mathbb{E}\int_0^T \left[\mathrm{Tr}(\Psi(\tau)Q^{\frac{1}{2}})(\Psi(\tau)Q^{\frac{1}{2}})^*\right] d\tau\right\}^{\frac{1}{2}}.$$

If $\Psi \in \mathcal{N}^2(0,T;L_2^0)$, then the integral $\int_0^t \Psi(\tau)\, dW(\tau)$ has sense.

In this paper, we use the so-called resolvent approach to the Volterra equation (1) (for details we refer to [10]).

Definition 1 *A family $(S(t))_{t\geq 0}$ of bounded linear operators in a Banach space B is called* `resolvent` *for (1) if the following conditions are satisfied:*

1. *$S(t)$ is strongly continuous on $\mathbb{R}_+$ and $S(0) = I$;*
2. *$S(t)$ commutes with the operator A:*
   $S(t)(D(A)) \subset D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \geq 0$;
3. *the following* `resolvent equation` *holds*

$$S(t)x = x + \int_0^t a(t-\tau)A\, S(\tau)x\, d\tau \tag{2}$$

for all $x \in D(A),\ t \geq 0$.

Let us emphasize that the family $(S(t))_{t\geq 0}$ does not create any semigroup and that $S(t),\ t\geq 0$, are generated by the pair $(A,a(t))$, that is, the operator A and the kernel function $a(t),\ t\geq 0$.

A consequence of the strong continuity of $S(t)$ is that $\sup_{t\leq T}\ ||S(t)|| < +\infty$ for any $T\geq 0$.

Definition 2 *We say that the function $a\in L^1(0,T)$ is* `completely positive` *on $[0,T]$, if for any $\mu\geq 0$, the solutions of the equations*

$$s(t)+\mu(a\star s)(t) = 1 \quad \textit{and} \quad r(t)+\mu(a\star r)(t) = a(t) \tag{3}$$

satisfy $s(t)\geq 0$ and $r(t)\geq 0$ on $[0,T]$.

The class of completely positive kernels, introduced in [2], arise naturally in applications, see [10].

Definition 3 *Suppose $S(t),\ t\geq 0$, is a resolvent. $S(t)$ is called* `exponentially bounded` *if there are constants $M\geq 1$ and $\omega\in\mathbb{R}$ such that*

$$||S(t)|| \leq M\, e^{\omega t}, \quad \textit{for all} \ \ t\geq 0\ ;$$

(M,ω) is called a `type` *of $S(t)$.*

Let us note that in contrary to C_0-semigroups, not every resolvent family needs to be exponentially bounded; for counterexamples we refer to [3].

In the paper, the key role is played by the following, yet non-published, result providing a convergence of resolvents.

Theorem 1 *Let A be the generator of a C_0-semigroup in B and suppose the kernel function a is completely positive. Then (A,a) admits an exponentially bounded resolvent $S(t)$. Moreover, there exist bounded operators A_n such that (A_n,a) admit resolvent families $S_n(t)$ satisfying $||S_n(t)|| \leq Me^{w_0 t}$ $(M\geq 1,\ w_0\geq 0)$ for all $t\geq 0,\ n\in\mathbb{N}$, and*

$$S_n(t)x \to S(t)x \quad \textit{as} \quad n\to +\infty \tag{4}$$

for all $x\in B,\ t\geq 0$.

Additionally, the convergence is uniform in t on every compact subset of $\mathbb{R}_+$.

Remark 1 *(a) The convergence (4) is an extension of the result due to Clément & Nohel [2]. The operators $A_n,\ n\in\mathbb{N}$, are the Yosida approximation of the operator A. For more details and the proof we refer to [7].*

(b) The above theorem give a partial answer to the following open problem for a resolvent family $S(t)$ generated by a pair (A,a): do exist bounded linear operators A_n generating resolvent families $S_n(t)$ such that $S_n(t)x\to S(t)x$?. Note that in case $a(t)\equiv 1$ the answer is yes, namely A_n are provided by the Hille-Yosida approximation of A and $S_n(t)=e^{tA_n}$.

2 Probabilistic results

In the sequel we shall use the following `Probability Assumptions`, abbr. (PA):

1. X_0 is an H-valued, $\mathcal{F}_0$-measurable random variable;
2. $\Psi \in \mathcal{N}^2(0,T;L_2^0)$ and the interval $[0,T]$ is fixed.

The following types of definitions of solutions to (1) are possible, see [6].

Definition 4 *Assume that (PA) hold. An H-valued predictable process $X(t)$, $t \in [0,T]$, is said to be a* `strong solution` *to (1), if X takes values in $D(A)$, P-a.s.,*

$$\int_0^T |a(T-\tau)AX(\tau)|_H \, d\tau < +\infty, \quad P-\text{a.s.} \tag{5}$$

and for any $t \in [0,T]$ the equation (1) holds P-a.s.

Let A^* be the adjoint of A with a dense domain $D(A^*) \subset H$ and the graph norm $|\cdot|_{D(A^*)}$.

Definition 5 *Let (PA) hold. An H-valued predictable process $X(t)$, $t \in [0,T]$, is said to be a* `weak solution` *to (1), if $P(\int_0^t |a(t-\tau)X(\tau)|_H d\tau < +\infty) = 1$ and if for all $\xi \in D(A^*)$ and all $t \in [0,T]$ the following equation holds*

$$\langle X(t), \xi\rangle_H = \langle X_0, \xi\rangle_H + \langle \int_0^t a(t-\tau)X(\tau)\, d\tau, A^*\xi\rangle_H$$
$$+\langle \int_0^t \Psi(\tau) dW(\tau), \xi\rangle_H, \quad P-\text{a.s.}$$

Definition 6 *Assume that X_0 is $\mathcal{F}_0$-measurable random variable such that $P(X_0 \in D(A)) = 1$. An H-valued predictable process $X(t)$, $t \in [0,T]$, is said to be a* `mild solution` *to the stochastic Volterra equation (1), if $\mathbb{E}(\int_0^t |S(t-\tau)\Psi(\tau)|^2_{L_2^0}\, d\tau) < +\infty$ for $t \le T$ and, for arbitrary $t \in [0,T]$,*

$$X(t) = S(t)X_0 + \int_0^t S(t-\tau)\Psi(\tau)\, dW(\tau), \quad P-\text{a.s.} \tag{6}$$

First, let us consider the case when W is an H-valued genuine Wiener process. In this case the equation (1) reads

$$X(t) = X_0 + \int_0^t a(t-\tau)\, AX(\tau)\, d\tau + W(t), \quad t \ge 0\, . \tag{7}$$

We define the convolutions:

$$W_S(t) := \int_0^t S(t-\tau)\, dW(\tau)$$
$$W_{S_n}(t) := \int_0^t S_n(t-\tau)\, dW(\tau),$$

where $S(t),\ S_n(t),\ t \geq 0$, are resolvents corresponding to A and A_n, respectively.

Now, we can recall several results from [7], not published.

Theorem 2 *Let A be the generator of a C_0-semigroup in H. Suppose the kernel function a is completely positive and $\mathrm{Tr}\, Q < +\infty$. Then*

$$\lim_{n\to+\infty} \mathbb{E}(\sup_{t\in[0,T]} |W_S(t) - W_{S_n}(t)|_H^p) = 0$$

for any $p \geq 2$.

Lemma 1 *Assume that A is the generator of a C_0-semigroup in H, the kernel function a is completely positive and $\mathcal{R}(S(t)) \subset D(A)$. If $X_0 = 0$, then the convolution $W_S(t)$ fulfills (7).*

Theorem 3 *Assume that A is the generator of a C_0-semigroup and the kernel function a is completely positive. Let $\mathcal{R}(S(t)) \subset D(A)$ for all $t > 0$ and $X_0 = 0$. Then the equation (7) has a strong solution. Precisely, the convolution $W_S(t)$ is the strong solution to (7).*

Theorem 3 is an extension of the semigroup case. It provides sufficient conditions for mild solutions to be strong solutions.

Now, let us consider the case when W is a cylindrical Wiener process. We define the convolution

$$W^\Psi(t) := \int_0^t S(t-\tau)\Psi(\tau)\, dW(\tau)$$

for $\Psi \in \mathcal{N}^2(0,T;L_2^0)$.

Proposition 1 *If $\Psi \in \mathcal{N}^2(0,T;L_2^0)$ and $\Psi(\cdot,\cdot)(U_0) \subset D(A)$, P-a.s., then the stochastic convolution W^Ψ fulfills the equation*

$$\langle W^\Psi(t), \xi\rangle_H = \int_0^t \langle a(t-\tau)W^\Psi(\tau), A^*\xi\rangle_H + \int_0^t \langle \xi, \Psi(\tau)dW(\tau)\rangle_H$$

for any $t \in [0,T]$ and $\xi \in D(A^*)$.

Proposition 1 (see [6]) enables to formulate the following results.

Proposition 2 *Let A be the generator of C_0-semigroup in H and suppose that the function a is completely positive. If Ψ and $A\Psi$ belong to $\mathcal{N}^2(0,T;L_2^0)$ and in addition $\Psi(\cdot,\cdot)(U_0) \subset D(A)$, P-a.s., then the following equality holds*

$$W^\Psi(t) = \int_0^t a(t-\tau)A\,W^\Psi(\tau)\,d\tau + \int_0^t \Psi(\tau)\,dW(\tau)\ .$$

Theorem 4 *Suppose that assumptions of Proposition 2 hold. Then the equation (1) has a strong solution. Precisely, the convolution W^Ψ is the strong solution to (1).*

3 Fractional Volterra equations

Let us note that the condition $\mathcal{R}(S(t)) \subset D(A),\ t \geq 0$ used in Theorem 3, is satisfied by a large class of resolvents. Particularly, when the equation (7) is parabolic in the sense of [10] and $a(t)$ is k-regular, e.g. $a(t) = t^{\alpha-1}/\Gamma(\alpha)$, $t \geq 0$, $\alpha \in (0,2)$, where Γ is the gamma function. This fact leads us to fractional Volterra equation of the following form

$$X(t) = X_0 + \int_0^t a(t-\tau)AX(\tau)d\tau + \int_0^t \Psi(\tau)\,dW(\tau), \quad t \geq 0, \tag{8}$$

when $a(t) = g_\alpha(t), \quad \alpha > 0$ with $g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$. Observe that for $\alpha \in (0,1]$, g_α are completely positive, but for $\alpha > 1$, g_α are not completely positive.

Now, the pairs $(A, g_\alpha(t))$ generate α-times resolvents $S_\alpha(t),\ t \geq 0$; for more details, see [1].

Remark 2 *Observe that the α-times resolvent family corresponds to a C_0-semigroup in case $\alpha = 1$ and a cosine family in case $\alpha = 2$. In consequence, when $1 < \alpha < 2$ such resolvent families interpolate C_0-semigroups and cosine functions. In particular, for $A = \Delta$, the integrodifferential equation corresponding to such resolvent family interpolates the heat equation and the wave equation, see, e.g. [4].*

We consider two cases:

(A1) A is the generator of C_0-semigroup and $\alpha \in (0,1)$;

(A2) A is the generator of a strongly continuous cosine family and $\alpha \in (0,2)$.

In this part of the paper, the results concerning a weak convergence of α-times resolvents play the key role. Using the very recent result due to Li and Zheng [8], we can formulate the approximation theorems for fractional Volterra equations.

Theorem 5 *Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ in a Banach space B such that*

$$\|T(t)\| \leq Me^{\omega t}, \quad t \geq 0\ . \tag{9}$$

Then, for each $0 < \alpha < 1$ there exist bounded operators A_n and α-times resolvent families $S_{\alpha,n}(t)$ for A_n satisfying $||S_{\alpha,n}(t)|| \leq MCe^{(2\omega)^{1/\alpha}t}$, for all $t \geq 0,\ n \in \mathbb{N}$, and

$$S_{\alpha,n}(t)x \to S_\alpha(t)x \quad as \quad n \to +\infty \tag{10}$$

for all $x \in B,\ t \geq 0$. Moreover, the convergence is uniform in t on every compact subset of $\mathbb{R}_+$.

Outline of the proof: The first assertion follows from [1, Theorem 3.1], that is, for each $0 < \alpha < 1$ there is an α-times resolvent family $(S_\alpha(t))_{t\geq 0}$ for A given by

$$S_\alpha(t)x = \int_0^\infty \varphi_{t,\alpha}(s)T(s)xds, \quad t > 0,$$

where $\varphi_{t,\gamma}(s) := t^{-\gamma}\Phi_\gamma(st^{-\gamma})$ and $\Phi_\gamma(z)$ is the Wright function defined as

$$\Phi_\gamma(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\,\Gamma(-\gamma n + 1 - \gamma)}, \quad 0 < \gamma < 1.$$

Define

$$A_n := nAR(n, A) = n^2R(n, A) - nI, \qquad n > w,$$

the *Yosida approximation* of A.

Since each A_n is bounded, it follows that for each $0 < \alpha < 1$ there exists an α-times resolvent family $(S_{\alpha,n}(t))_{t\geq 0}$ for A_n given as

$$S_{\alpha,n}(t) = \int_0^\infty \varphi_{t,\alpha}(s)e^{sA_n}ds, \quad t > 0.$$

We recall that the Laplace transform of the Wright function corresponds to $E_\gamma(-z)$ where E_γ denotes the Mittag-Leffler function. In particular, $\Phi_\gamma(z)$ is a probability density function. It follows that for $t \geq 0$:

$$\begin{aligned}
\|S_{\alpha,n}(t)\| &\leq \int_0^\infty \varphi_{t,\alpha}(s)\|e^{sA_n}\|ds \\
&\leq M\int_0^\infty \varphi_{t,\alpha}(s)e^{2\omega s}ds = M\int_0^\infty \Phi_\alpha(\tau)e^{2\omega t^\alpha \tau}d\tau = ME_\alpha(2\omega t^\alpha).
\end{aligned}$$

The continuity in $t \geq 0$ of the Mittag-Leffler function and its asymptotic behavior, imply that for $\omega \geq 0$ there exists a constant $C > 0$ such that

$$E_\alpha(\omega t^\alpha) \leq Ce^{\omega^{1/\alpha}t}, \quad t \geq 0,\ \alpha \in (0, 2).$$

This gives

$$\|S_{\alpha,n}(t)\| \leq MCe^{(2\omega)^{1/\alpha}t}, \quad t \geq 0.$$

Now we recall the fact that $R(\lambda, A_n)x \to R(\lambda, A)x$ as $n \to \infty$ for all λ sufficiently large (e.g. [9, Lemma 7.3]), so we can conclude from [8, Theorem 4.2] that

$$S_{\alpha,n}(t)x \to S_\alpha(t)x \quad \text{as} \quad n \to +\infty$$

for all $x \in B$, uniformly for t on every compact subset of $\mathbb{R}_+$ $\square$

An analogous convergence for α-times resolvents can be proved in another case, too.

Theorem 6 *Let A be the generator of a C_0-cosine family $(T(t))_{t\geq 0}$ in a Banach space B. Then, for each $0 < \alpha < 2$ there exist bounded operators A_n and α-times resolvent families $S_{\alpha,n}(t)$ for A_n satisfying $||S_{\alpha,n}(t)|| \leq MCe^{(2\omega)^{1/\alpha}t}$, for all $t \geq 0$, $n \in \mathbb{N}$, and $S_{\alpha,n}(t)x \to S_\alpha(t)x$ as $n \to +\infty$ for all $x \in B$, $t \geq 0$. Moreover, the convergence is uniform in t on every compact subset of $\mathbb{R}_+$.*

Now, we are able to formulate the results analogous to that in section 2.

Theorem 7 *Assume that* **(A1)** *or* **(A2)** *holds and $\mathcal{R}(S_\alpha(t)) \subset D(A)$. If $X_0 = 0$, then the equation (7) has a strong solution.*

Outline of the proof: We define the convolutions

$$W_{S_\alpha}(t) := \int_0^t S_\alpha(t-\tau)\, dW(\tau), \tag{11}$$

$$W_{S_{\alpha,n}}(t) := \int_0^t S_{\alpha,n}(t-\tau)\, dW(\tau), \tag{12}$$

where $S_\alpha(t), S_{\alpha,n}(t),\ t \geq 0$, are resolvents corresponding to the operators A and A_n, respectively.

By [6, Corollary 1], for every $n \in \mathbb{N}$ the convolution $W_{S_{\alpha,n}}(t)$ fulfills (7), where $W_{S_{\alpha,n}}(t)$ and A_n are as above. Because Theorem 2 holds for the convolutions $W_{S_{\alpha,n}}(t)$ and $W_{S_\alpha}(t)$, we can assume that $W_{S_{\alpha,n}}(t) \to W_{S_\alpha}(t)$ as $n \to +\infty$, P-a.s.

By the Lebesgue dominated convergence theorem

$$\lim_{n\to+\infty} \sup_{t\in[0,T]} \mathbb{E}|W_{S_{\alpha,n}}(t) - W_{S_\alpha}(t)|_H^2 = 0. \tag{13}$$

From assumptions $\mathcal{R}(S_\alpha(t)) \subset D(A)$, so $P(W_{S_\alpha}(t) \in D(A)) = 1$.

Using (13) and the fact that $\lim_{n\to\infty} A_n x = Ax$ for any $x \in D(A)$, we have

$$\lim_{n\to+\infty} \sup_{t\in[0,T]} \mathbb{E}|A_n W_{S_{\alpha,n}}(t) - AW_{S_\alpha}(t)|_H^2 = 0.$$

Hence, $W_{S_\alpha}(t) = \int_0^t g_\alpha(t-\tau)\, AW_{S_\alpha}(\tau)\, d\tau + W(t),\ t \in [0,T]$, P-a.s.

Because the convolution $W_{S_\alpha}(t)$ has integrable trajectories (see [6]) and the closed linear unbounded operator A becomes bounded on $D(A)$ endowned with the norm $|\cdot|_{D(A)}$ (see [11, Chapter 5]), then $A\,W_{S_\alpha}(\cdot) \in L^1([0,T];H)$. Hence, the function $g_\alpha(T-\tau)\,A\,W_{S_\alpha}(\tau)$ is integrable with respect to τ, what finishes the proof. $\square$

Theorem 8 *Assume that* **(A1)** *or* **(A2)** *holds. If Ψ and $A\Psi$ belong to $\mathcal{N}^2(0,T;L_2^0)$ and in addition $\Psi(\cdot,\cdot)(U_0) \subset D(A)$, P-a.s., then the equation (1) with $X_0 = 0$ has a strong solution. Precisely, the convolution*

$$W_\alpha^\Psi(t) := \int_0^t S_\alpha(t-\tau)\,\Psi(\tau)\,dW(\tau)$$

is the strong solution to (1).

Outline of the proof: First, analogously like in [6], we show that the convolution $W_\alpha^\Psi(t)$ fulfills the following equation

$$W_\alpha^\Psi(t) = \int_0^t g_\alpha(t-\tau) A\, W_\alpha^\Psi(\tau)\, d\tau + \int_0^t \Psi(\tau)\, dW(\tau)\,. \tag{14}$$

Next, we have to show the condition

$$\int_0^T |g_\alpha(T-\tau) AW_\alpha^\Psi(\tau)|_H\, d\tau < +\infty, \quad P-a.s., \quad \alpha > 0, \tag{15}$$

that is, the condition (5) adapted for the fractional Volterra equation (8).

The convolution $W_\alpha^\Psi(t)$ has integrable trajectories (see [6]), that is, $W_\alpha^\Psi(\cdot) \in L^1([0,T];H)$, P-a.s. The closed linear unbounded operator A becomes bounded on $(D(A), |\cdot|_{D(A)})$, see [11, Chapter 5]. Hence, $AW_\alpha^\Psi(\cdot) \in L^1([0,T];H)$, P-a.s. Therefore, the function $g_\alpha(T-\tau)AW_\alpha^\Psi(\tau)$ is integrable with respect to τ, what completes the proof. $\square$

4 Examples

The class of equations fulfilling our conditions depends on where the operator A is defined, in particular, the domain of A depends on each considered problem, and also depends on the properties of the kernel function a.

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Consider the differential operator of order $2m$:

$$A(x,D) = \sum_{|\alpha|\leq 2m} a_\alpha(x) D^\alpha \tag{16}$$

where the coefficients $a_\alpha(x)$ are sufficiently smooth complex-valued functions of x in $\overline{\Omega}$. The operator $A(x,D)$ is called `strongly elliptic` if there exists a constant $c > 0$ such that

$$Re(-1)^m \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha \geq c|\xi|^{2m} \tag{17}$$

for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^n$.

Let $A(x,D)$ be a given strongly elliptic operator on a bounded domain $\Omega \subset \mathbb{R}^n$ and set $D(A) = H^{2m}(\Omega) \cap H_0^m(\Omega)$. For every $u \in D(A)$ define

$$Au = A(x,D)u. \tag{18}$$

Then the operator $-A$ is the infinitesimal generator of an analytic semigroup of operators on $H = L^2(\Omega)$ (cf. [9, Theorem 7.2.7]). We note that if the operator A has constant coefficients, the result remains true for the domain $\Omega = \mathbb{R}^n$.

A concrete example is the Laplacian

$$\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}, \tag{19}$$

since $-\Delta$ is clearly strongly elliptic. It follows that Δu on $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ is the infinitesimal generator of an analytic semigroup on $L^2(\Omega)$.

In particular, by [10, Corollary 2.4] the operator A given by (16) generates an analytic resolvent $S(t)$ whenever $a \in C(0,\infty) \cap L^1(0,1)$ is completely monotonic. As a consequence $\mathcal{R}(S(t)) \subset D(A)$ for all $t > 0$.

This example fits in our results (Theorem 3) if a is also completely positive. For example: $a(t) = t^{\alpha-1}/\Gamma(\alpha)$ is both, completely positive and completely monotonic for $0 < \alpha \leq 1$ (but not for $\alpha > 1$).

Another class of examples is provided by the following: suppose $a \in L^1_{loc}(\mathbb{R}_+)$ is of subexponential growth and $\pi/2$-sectorial, and let A generate a bounded analytic C_0-semigroup in a complex Hilbert space H. Then it follows from [10, Corollary 3.1] that the Volterra equation of scalar type $u = a * Au + f$ is parabolic. If, in addition, $a(t)$ is k-regular for all $k \geq 1$ we obtain from [10, Theorem 3.1] the existence of a resolvent $S \in C^{k-1}((0,\infty), \mathcal{B}(H))$ such that $\mathcal{R}(S(t)) \subset D(A)$ for $t > 0$ (see [10, p.82 (f)]). These observations together with Theorem 3 give us the following result.

Corollary 1 *Suppose that A generates a bounded analytic C_0-semigroup in a complex Hilbert space H and $a \in L^1_{loc}(\mathbb{R}_+)$ is of subexponential growth, $\pi/2$-sectorial, completely*

positive and k-regular for all $k \geq 1$. Then the equation

$$X(t) = X_0 + \int_0^t a(t-\tau)\, AX(\tau)\, d\tau + W(t), \quad t \geq 0\,.$$

has a strong solution.

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